

# Universal Scaling in Saddle-Node Bifurcation Cascades (II)

## Intermittency Cascade

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### Abstract

The presence of saddle-node bifurcation cascade in the logistic equation is associated with an intermittency cascade; in a similar way as a saddle-node bifurcation is associated with an intermittency. We merge the concepts of bifurcation cascade and intermittency. The mathematical tools necessary for this process will describe the structure of the Myrberg-Feigenbaum point.

*Key words:* Intermittency Cascade. Saddle-Node bifurcation cascade. Attractor of attractors. Structure of Myrberg-Feigenbaum points.

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### 1 Introduction

Period-doubling [1,2], Quasi-periodicity [3,4] and Intermittency [5,6] are well known routes of transition from periodic to chaotic behavior, and whose origin is in local bifurcations. Initially, the system has a stable limit cycle, for a range control parameter  $r$ . As this parameter is increased beyond a critical value  $r_c$  system behavior changes according to a local bifurcation that occurs at  $r_c$ .

In order to study the genesis of the transition we resort to the Poincare section. In this section, the original stable limit cycle of the system generates a fixed point, whose evolution is studied in parallel to the control parameter changes.

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Paying attention to the different local bifurcation kinds we will have different transitions to chaos. So, if the fixed point shows successive pitchfork bifurcations, which double repeatedly the period of the original orbit, the Feigenbaum period doubling cascade is obtained. The final ending is a period- $\infty$  orbit, a chaotic attractor.

Quasi-periodicity occurs as a new Hopf bifurcation generates a second frequency in the system, which is incommensurate with the original system frequency. If an irrational winding number is fixed it goes through chaos.

At last, the intermittency is a chaotic regime characterized by an apparently regular behavior, which undergoes irregular bursts from time to time. This intermittency between two behaviors names this kind of chaos.

The regular behavior or laminar regime corresponds to an evolution of the system in a narrow region or a channel in the phase space. Such regular behavior stems from the fact that the system maintains a “ghost” of laminar regime

Whereas in the other transitions to chaos, Period-doubling and Quasi-periodicity, the system is totally regular before the transition and chaotic later. This does not happen with the intermittency. The intermittency shows a continue transition from regular behavior to a chaotic one. The smaller the value  $\varepsilon = r - r_c$  is the longer the laminar regime will be and the lesser it will be altered by a chaotic behavior, due to the fact that the average time of laminar regime  $\langle l \rangle$  for small  $\varepsilon$  is  $\varepsilon^{-\beta}$ , being  $\beta > 0$ . Therefore, beyond  $r_c$  the laminar behavior alternates with irregular bursts, the smaller (bigger) the value  $\varepsilon$  is the more (less) the laminar regime  $\langle l \rangle$  is.

The intermittency transition was discussed by Pomeau and Manneville [5,6]. They pointed out three kinds of intermittency. The system is periodic for parameter values smaller than the critical point  $r < r_c$ . This periodic behavior generates a stable fixed point in the Poincare section. As the parameter reaches the critical value  $r = r_c$  the fixed point loses its stability. The loss of stability is caused when the eigenvalue modulus of the linearized Poincare map becomes larger than unit. This may happen in three different ways.

- i) There is a real eigenvalue crossing the unit circle by plus one. A saddle-node bifurcation is generated, which has associated type-I intermittency. In this case the average length of laminar regime time is  $\langle l \rangle \sim \varepsilon^{-\frac{1}{2}}$ .
- ii) A couple of complex conjugate eigenvalues crosses the unit circle. This circumstance is associated with the birth of a Hopf bifurcation and it involves a type-II intermittency with  $\langle l \rangle \sim \varepsilon^{-1}$ .
- iii) A real eigenvalue crosses the unit circle by minus one, generating a flip bifurcation. This time the type-III intermittency occurs and we obtain  $\langle l \rangle \sim$

$\varepsilon^{-1}$ .

Other kinds of intermittencies have been studied such as type-X [7] one which shows a transition with hysteresis and type-V one [8] for discontinuous maps. Intermittencies have also been studied whose laminar regime occurs alternatively between two channels, as a result of symmetry in the problem [9].

The generalization and logical development of this latter problem is to find intermittencies with an arbitrary number of channels. Furthermore it is desirable to have a different number of channels for different values of control parameters in the same system, and not to look for different systems *ad hoc* with the appropriate symmetries which show a fixed number of channels. We say that such behavior is desirable in a unique system because, if a change of the control parameter implies an increase of the number of channel, what is obtained is an intermittency cascade; in a similar way a change of the control parameter in the logistic map generates a period doubling cascade.

In this paper we are going to show that such phenomenon occurs in the logistic map, benefiting from the fact that this map shows saddle-node bifurcation cascades [10] and that the type-I intermittency is associated with the saddle-node bifurcation. We will characterize control parameter values at which successive intermittencies are generated, how some intermittencies are related to others, the number of channels, which is the average time of laminar regime of intermittencies, and what relationship there is between such regimes for different intermittencies. To answer these questions we will use the universal properties of the logistic map [1,2] and the saddle-node bifurcation cascade this map shows [10].

The saddle-node bifurcation cascade is a sequence of saddle-node bifurcations in which the number of fixed points showing this kind of bifurcation is duplicated. The successive elements of the sequence are given by an equation identical to the one that Feigenbaum found for period doubling cascade [10].

The way of acting is as follows. As we mentioned above type-I intermittency is associated to one saddle-node bifurcation. Therefore, each element of the sequence of the saddle-node bifurcation cascade has associated a type-I intermittency. The number of channels of this intermittency coincides with the number of fixed points that simultaneously show a saddle-node bifurcation. For instance, the saddle-node bifurcation cascade, symbolized by the sequence  $3, 3 \cdot 2, 3 \cdot 2^2, 3 \cdot 2^3, \dots, 3 \cdot 2^q$ , points out that there are 3 fixed points at a first parameter value  $r = r_3$ , there are  $3 \cdot 2$  at  $r = r_{3 \cdot 2}$  and so on. Each saddle-node fixed point contributes to the intermittency with one channel, accordingly in this intermittency cascade there will be a sequence of intermittencies with  $3, 3 \cdot 2, 3 \cdot 2^2, 3 \cdot 2^3, \dots, 3 \cdot 2^q, \dots$  channels.

The channels responsible for laminar regime are close to critical points of lo-

gistic map (Fig. .2). The way, how the neighborhood of these critical points are contracted in the successive iterated of the map, determines how the channels are contracted and it allows us to look for the connection between them. The scaling of the neighborhood of critical points for iterated map was calculated near a pitchfork bifurcation by Feigenbaum [1]. We will follow this work to calculate the scaling near a saddle-node bifurcation, because it is here where intermittency occurs.

As many iterated one-dimensional maps are nearly quadratic under renormalization [11], we have to expect that intermittencies cascade is a common phenomenon in many natural processes.

Let be the logistic equation  $x_{n+1} = f(x_n) = r x_n(1 - x_n)$ . The graph of  $f^3$  ( $f^n = f \circ \dots \circ f$ ) is shown in Fig. .1, where it is tangent to line  $x_{n+1} = x_n$ , at  $r = r_c = 1 + \sqrt{8}$ , which means a period-3 orbit exists. There are three saddle-node fixed points, and we are at the genesis of a saddle-node bifurcation. For  $r > r_c$  the valleys and the hills of  $f^3$  are sharper than at  $r = r_c$ , and each saddle-node point has generated two points: one saddle and one node. If we decreased from  $r > r_c$  to  $r < r_c$  we would observe that the saddle point approaches the node one, touching at  $r = r_c$ , as the saddle-node bifurcation occurs. For  $r < r_c$  the valleys and the hills are pulled away from the diagonal and saddle-node fixed points disappear. After the bifurcations have disappeared three narrow channels, delimited by the graph of  $f^3$  and the diagonal, remain, which are responsible for laminar regime of intermittency. This is what we meant when we said earlier that each S-N point would be responsible for generating a channel in the intermittency cascade.

Each iterated of logistic map will take a long time to go through these channels (see Fig. .2). The average number of iterated inside a channel is given by  $\langle l \rangle \sim \varepsilon^{-\frac{1}{2}}$  [6], and so it is for the set of three channels.

The saddle-node bifurcation cascade involves saddle-node bifurcations with  $q, q \cdot 2, q \cdot 2^2, q \cdot 2^3, \dots, q \cdot 2^n$ ,  $q \neq 2^m$  fixed points for the maps  $f^q, f^{2q}, \dots, f^{q2^n}$  respectively, and the same number of channels for the intermittency. Fig. .3 and .4 show saddle-node bifurcations for  $f^{3 \cdot 2} = f^6$  and  $f^{3 \cdot 2^2} = f^{12}$  respectively.

We want to connect the length of laminar regime  $\langle l \rangle$  of one intermittency with the laminar regime of other intermittencies present in the cascade. If we notice in the neighborhood of point  $(\frac{1}{2}, \frac{1}{2})$  in Fig. .3 we will see that the graph of  $f^3$  is reproduced in Fig. .1 at  $r = r_c$ , escalated by a factor  $\frac{1}{\alpha}$ ,  $\alpha > 1$ . The same can be said in the neighborhood of point  $(\frac{1}{2}, \frac{1}{2})$  in Fig. .4. As the iterated  $f^{q2^n}$  with  $n \rightarrow \infty$  are considered the constant  $\alpha$  we get is the Feigenbaum constant. (see appendix)

If we come back to Fig. .3 we will notice that  $f^{3 \cdot 2}$  reproduces again the graph

of  $f^3$  in the neighborhood of  $(f(\frac{1}{2}), f(\frac{1}{2}))$ , and that this one is not escalated by the same factor as in the neighborhood of the point  $(\frac{1}{2}, \frac{1}{2})$ . From one iterated to the following one, that is, from  $f^{3 \cdot 2^n}$  to  $f^{3 \cdot 2^{n+1}}$ , half of the neighborhood of critical points escalate approximately with  $\frac{1}{\alpha}$  and the other half with  $\frac{1}{\alpha^2}$  (see appendix). We will be able to answer our questions because  $f^3$  is reproduced in the neighborhood of critical points of  $f^{3 \cdot 2^n}$  and also because we know how these neighborhoods scale in the successive elements of the cascade

## 2 Intermittency Cascade

Let  $r_{q \cdot 2^n, SN}$  be the parameter value at which  $f^{q \cdot 2^n}$ ,  $q \neq 2^m$  has a saddle-node bifurcation, that is,  $f^{q \cdot 2^n}$  has a saddle-node orbit with  $q \cdot 2^n$  points. The points of this orbit are located right where the function  $f^{q \cdot 2^n}$  is tangent to the line  $y = x$ . The  $q \cdot 2^n$  points can be classified into  $2^n$  sets, each one of them having  $q$  points. The  $2^n$  sets correspond to  $2^n$  critical points of  $f^{2^n}$  closest to the line  $y = x$ . The  $q$  saddle-node points are captured in a neighborhood of each one of these critical points, in other words, the graph of  $f^q$  is captured in every one of such neighborhoods; for instance, in  $f^{3 \cdot 2^2}$  we notice how the graph of  $f^3$  is captured in the neighborhoods of the  $2^2$  critical points of  $f^{2^2}$  (Fig. .4).

If we choose

$$r = r_{q \cdot 2^n, SN} - \varepsilon \quad 0 < \varepsilon \ll 1 \quad (1)$$

then the saddle-node bifurcation will be about to occur. In these conditions, there are  $q \cdot 2^n$  points where  $f^{q \cdot 2^n}$  is almost tangent to the line  $y = x$ . In such points there are  $q \cdot 2^n$  channels, which are formed by the graph of  $f^{q \cdot 2^n}$  and the line  $y = x$ . These channels are the narrower the smaller  $\varepsilon$  is in Eq. (1), and in them the laminar regime occurs. The time to cross the channel depends on  $\varepsilon$ .

Let  $\langle l \rangle_n$  be the average time that the iterates of  $x_{n+1} = f(x_n)$  spend to cross the  $q \cdot 2^n$  channels generated by  $f^{q \cdot 2^n}$ . If we consider the laminar regime of an intermittency of  $f^{q \cdot 2^{n+1}}$  then the number of channels will become duplicated because  $f^{q \cdot 2^{n+1}} = f^{q \cdot 2^n} \circ f^{q \cdot 2^n}$ , in other words, the graph of  $f^q$  is duplicated close to the critical points of  $f^{2^n}$ . But in the doubling process the replicas of graph of  $f^q$  are contracted, half of them as  $\frac{1}{\alpha}$  and the others as  $\frac{1}{\alpha^2}$  as  $n \rightarrow \infty$ , where  $\alpha$  is the Feigenbaum constant (see appendix). Accordingly, for  $\varepsilon = r_{q \cdot 2^{n+1}, SN} - r$  the intermittency of  $f^{q \cdot 2^{n+1}}$  shows the channels of the intermittency of  $f^{q \cdot 2^n}$  duplicated, but half of them contracted as  $\frac{1}{\alpha}$  and the other contracted as  $\frac{1}{\alpha^2}$ . As the average time for the intermittency of  $f^{q \cdot 2^n}$  is  $\langle l \rangle_n$  it turns out that the average time for the intermittency of  $f^{q \cdot 2^{n+1}}$  will be  $\frac{\langle l \rangle_n}{\alpha}$ , which comes from the channels contracted by  $\frac{1}{\alpha}$ , plus  $\frac{\langle l \rangle_n}{\alpha^2}$ , which are given

by the channels contracted by  $\frac{1}{\alpha^2}$ . In conclusion, the average time of laminar regime of the intermittency of  $f^{q \cdot 2^{n+1}}$  is  $\langle l \rangle_n (\frac{1}{\alpha} + \frac{1}{\alpha^2})$ , where  $\langle l \rangle_n$  is the average time of laminar regime of  $f^{q \cdot 2^n}$ , and both  $f^{q \cdot 2^n}$  and  $f^{q \cdot 2^{n+1}}$  are at the same distance  $\varepsilon$  from the corresponding saddle-node bifurcation in the parameter space, that is,  $r = r_{q \cdot 2^{n+1}, SN} - \varepsilon$  and  $r = r_{q \cdot 2^n, SN} - \varepsilon$ .

Let's notice that in a saddle-node bifurcation cascade the laminar regime from an intermittency to the next one in the sequence is decreased by a factor  $(\frac{1}{\alpha} + \frac{1}{\alpha^2})$ . Accordingly, the average time for intermittency of  $f^{q \cdot 2^{n+m}}$  is

$$\langle l \rangle_{n+m} = \langle l \rangle_n (\frac{1}{\alpha} + \frac{1}{\alpha^2})^m \quad (2)$$

Given the saddle-node bifurcation cascade of  $f^{q \cdot 2^n}, f^{q \cdot 2^{n+1}}, \dots, f^{q \cdot 2^{n+m}}, \dots$  if the bifurcation parameter of  $f^{q \cdot 2^n}$  is at  $r_{q \cdot 2^n, SN}$  then the other bifurcation parameters are given by [10]

$$r_{q \cdot 2^{n+1}, SN} = \frac{1}{\delta} r_{q \cdot 2^n, SN} + (1 - \frac{1}{\delta}) r_\infty \quad (3)$$

where  $\delta$  is the Feigenbaum constant; and  $r_\infty$  is the Myrberg-Feigenbaum point of a canonical window where Feigenbaum cascade finishes and where also all saddle-node bifurcation cascades finish, whatever  $q \neq 2^m$  is.

Eqs. (2) and (3) determine the intermittency cascade, because the parameter values at which it occurs and the average time of its corresponding laminar regimes are known.

The former results are valid if a intermittency cascade occurs in a period- $j$  window instead of a canonical window. Because both the scaling of laminar regime  $(\frac{1}{\alpha} + \frac{1}{\alpha^2})$  and Eq. (3) are valid in a period- $j$  window, although the Eq. (3) turns into

$$r_{q \cdot 2^{n+1}, SN} = \frac{1}{\delta} r_{q \cdot 2^n, SN} + (1 - \frac{1}{\delta}) r_{\infty, j}$$

to indicate that the convergence is the one to the Myrberg-Feigenbaum point  $r_{\infty, j}$  of the period- $j$  window.

There is a second way of changing the control parameter in the intermittency cascade, which is more important to the experimenters.

The Eq. (2) gives the length of average times to an intermittency cascade associated with the saddle-node bifurcation cascade of  $f^{q \cdot 2^n}, f^{q \cdot 2^{n+1}}, \dots, f^{q \cdot 2^{n+m}}, \dots$ , if the value of  $\varepsilon$  is constant. Such value gives the distance from the control

parameter to the saddle-node bifurcation parameter. Nonetheless, it is possible to change  $\varepsilon$  in the successive saddle-node bifurcations in such a way that the average time of laminar regimen is kept constant, and equal to a  $\langle l \rangle_n$ , for the whole intermittency cascade. For that purpose, all we need is to have the value  $\varepsilon$ , taken in the first intermittency, is rescaled by a factor  $(\frac{1}{\alpha} + \frac{1}{\alpha^2})$  for each one of the successive intermittencies of the cascade, that is, the values

$$\varepsilon, \varepsilon(\frac{1}{\alpha} + \frac{1}{\alpha^2}), \dots, \varepsilon(\frac{1}{\alpha} + \frac{1}{\alpha^2})^m, \dots \quad (4)$$

for  $m = 0, 1, 2, 3, \dots$ . This is so because

$$\langle l \rangle \propto \frac{1}{\varepsilon} \quad (5)$$

for the type-I intermittency, which is present in the logistic equation. If the values of (4) are introduced in Eq. (5) then the laminar regimen is increased in a factor which is the same as the one that contracts according to Eq. (2). The result is that the average time of laminar regimen stays constant.

It is necessary to change  $\varepsilon$  in this way. As Eq. (3) shows a geometric progression, of ratio  $\frac{1}{\delta}$ , if we held  $\varepsilon$  constant then very quickly the value  $r = r_{q \cdot 2^n, SN} - \varepsilon$  would not be within the parameter interval  $[r_{q \cdot 2^{n+m+1}, SN}, r_{q \cdot 2^{n+m}, SN}]$  and we would not observe channels corresponding to two successive saddle-node bifurcation of the cascade. The result would be that  $r \ll r_{q \cdot 2^{n+m+1}, SN}$  and the intermittency cascade would not be observed. Obviously, this is vital for the experimenters and for the development of numerical experiments as well.

Bear in mind that as  $\varepsilon$  changes as in Eq. (4) it turns out that also  $\varepsilon$  changes as geometric progression of ratio  $(\frac{1}{\alpha} + \frac{1}{\alpha^2})$ . This geometric progression converges faster than the progression Eq. (3). Accordingly, the parameter value  $r$  can be held such that  $r \in [r_{q \cdot 2^{n+m+1}, SN}, r_{q \cdot 2^{n+m}, SN}]$ , and therefore the channel associated with the saddle-node bifurcation at  $r_{q \cdot 2^{n+m}, SN}$  can be observed. It is necessary for the experimenter to change the parameter as in Eq. (4), in order to stay close to successive saddle-node bifurcations of the cascade and get a constant value of  $\langle l \rangle$ . It is easy to get this variation because the bifurcation parameters are given by Eq. (3).

### 3 Myrberg-Feigenbaum point structure

As shown in Fig. .1 we can see a saddle-node bifurcation of  $f^3$ . This same figure appears twice in Fig. .2. They are the two first elements of the saddle-node bifurcation cascade of  $f^{3 \cdot 2^n}$ . The bigger  $n$  is the more times Fig. .1 is

replicated in the graph of  $f^{3 \cdot 2^n}$  along the line  $y = x$ .

As shown in the appendix, Fig. .1 appears twice more every time we move on one stage in a saddle-node bifurcation cascade, half of the figures are contracted by  $\frac{1}{\alpha}$  and the other half by  $\frac{1}{\alpha^2}$ . The outcome is that in a saddle-node bifurcation cascade the points, that are tangent to the line  $y = x$ , duplicate at the same time as the region they are placed in contracts.

We would hope to find any kind of Cantor set and, what is worse, one Cantor set for each period- $q \cdot 2^n$ ,  $q \neq 2^m$  saddle-node bifurcation cascade, because all saddle-node bifurcation cascades approach the Myrberg-Feigenbaum point  $r_\infty$  as  $n \rightarrow \infty$ . Nonetheless the solution is extraordinary simple at the limiting value  $r_\infty$ .

If we consider the cascade  $q, q \cdot 2, \dots, q \cdot 2^n, \dots, q \neq 2^m$  it will turn out that the graph of  $f^q$  will be reproduced in the neighborhood of the critical points of  $f^{2^n}$ , which correspond to the points of the restricted-supercycle given by  $\{\frac{1}{2}, f(\frac{1}{2}), f^2(\frac{1}{2}), \dots, f^{2^n-1}(\frac{1}{2})\}$  (see appendix). Half of these neighborhoods are contracted by  $\frac{1}{\alpha}$  and the other half by  $\frac{1}{\alpha^2}$ , each time we move on one stage in saddle-node bifurcation cascade, that is, the saddle-node points duplicate. Therefore, in the limit  $n \rightarrow \infty$  each neighborhood has collapsed to a point of  $\{\frac{1}{2}, f(\frac{1}{2}), f^2(\frac{1}{2}), \dots, f^{2^n-1}(\frac{1}{2})\}_{n \rightarrow \infty}$ . The points of  $\{\frac{1}{2}, f(\frac{1}{2}), f^2(\frac{1}{2}), \dots, f^{2^n-1}(\frac{1}{2})\}_{n \rightarrow \infty}$  coincide with the period doubling orbit as  $n \rightarrow \infty$ . The outcome is that the period- $2^n$  ( $n \rightarrow \infty$ ) chaotic orbit coincides with period- $q \cdot 2^n$ ,  $q \neq 2^m$ ,  $n \rightarrow \infty$  orbit—in the sense of limit—that is, the limits cannot tell from each other. We have the same limit orbit, both the one which comes from period doubling cascade at  $r < r_\infty$ , and the one which comes from saddle-node bifurcation cascades at  $r > r_\infty$ .

The fact that the period- $q \cdot 2^n$  saddle-node orbit tends to the period- $2^n$  as  $n \rightarrow \infty$  hides another fact: the collapse of period- $q \cdot 2^n$  window to a point at Myrberg-Feigenbaum point  $r_\infty$ . This is so because the period- $q \cdot 2^n$  window starts with the birth of the period- $q \cdot 2^n$  saddle-node orbit and the period- $q \cdot 2^{n+1}$  window starts with the birth of the period- $q \cdot 2^{n+1}$  saddle-node. As the birth of both saddle-node orbits tends to  $r_\infty$ , as  $n \rightarrow \infty$ , then the window length tends to zero. This result is captured in the expression (see [10])

$$\frac{L_n}{L_{n+1}} = \delta$$

which shows that the length of two successive windows of a saddle-node bifurcation cascade are contracted by a factor  $\delta$ , being  $\delta$  Feigenbaum constant, that is, the windows length tends to zero.

If we consider the period- $q \cdot 2^n$  window it will have a Myrberg-Feigenbaum point



$r_{\infty, q \cdot 2^n}$ , at which its corresponding Feigenbaum cascade will finish. As  $n \rightarrow \infty$ , the window length tends to zero and it brings the following consequences and interpretation, relative to period- $q \cdot 2^n$  window.

i) The whole period doubling process also collapses to a point, and the same happens to the rest of the saddle-node bifurcation cascades present in the period- $q \cdot 2^n$  window, because the process occurs in a window whose length tends (collapses) to zero.

ii) The Myrberg-Feigenbaum point  $r_\infty$  of canonical window and the Myrberg-Feigenbaum point  $r_{\infty, q \cdot 2^n}$  of the period- $q \cdot 2^n$  window are the closer to each other the bigger  $n$  is. The distance tends to zero as  $n \rightarrow \infty$ . The same happens with every one of saddle-node bifurcation cascades located in the period- $q \cdot 2^n$  window. Accordingly the accumulating point of every saddle-node bifurcation cascade (a new Myrberg-Feigenbaum) tends to  $r_{\infty, q \cdot 2^n}$ , and therefore it tends to  $r_\infty$ . The process is applied again to the new windows which are born from a saddle-node bifurcation cascade and so forth.

This explains why a Myrberg-Feigenbaum point is an attractor of other Myrberg-Feigenbaum points, which are attractor of other Myrberg-Feigenbaum points and so forth (see [10])

The former convergence process has been expounded for a fixed saddle-node bifurcation cascade with  $q \neq 2^m$ , but it is valid for any value of  $q$ . Therefore there are infinite sequences which mimic the former process, one for each value of  $q \neq 2^m$ .

The approaching to the Myrberg-Feigenbaum point, and the multiplicity of convergent sequences, explain completely the mechanism of attractor of attractor introduced in [10]

## 4 CONCLUSIONS

The presence of saddle-node bifurcation cascade in the logistic equation assures the genesis of a intermittency cascade. Each saddle-node bifurcation of the bifurcation cascade is associated with an intermittency. As the location of the saddle-node bifurcation is known it brings that so is the genesis of the intermittency cascade. The knowledge, a priori, of the length of the laminar regime in the type-I intermittency, and of the scaling the peaks and valleys of the successive iterated of logistic map, allows us to establish the length of the laminar regime in the intermittency cascade.

The intermittency cascade is a phenomenon that takes place in all windows

of the logistic map, and not only in windows associated to first-occurrence orbits.

It is proved that the windows collapse to the Myrberg-Feigenbaum points, this mechanism being responsible for the fact that Myrberg-Feigenbaum points are attractors of attractors.

## Acknowledgments

The author wishes to thank Daniel Rodríguez-Pérez for helpful discussions and help in the preparation of the manuscript.

## APPENDIX

Let's find the scaling law of high-order cycles in the saddle-node bifurcation cascade. To do so we will follow the Feigenbaum work [12] , which is carried out close to pitchfork bifurcation.

The scaling law is not determined by the location of the elements on the x-axis, but by their order as iterates of  $x = \frac{1}{2}$  ( or of  $x = 0$  after a coordinate translation that moves  $x = \frac{1}{2}$  to  $x = 0$ ). This is the point which necessarily belongs to any supercycle. Feigenbaum denotes the distance from the m-th element of a  $2^n$ -supercycle to its nearest neighbor by

$$d_n(m) = x_m - f_{R_n}^{2^{n-1}}(x_m)$$

where  $R_n$  is the control parameter value at which supercycle occurs.

To generalize the latter definition to the period- $q \cdot 2^n$  saddle-node orbit, which is in the saddle-node bifurcation cascade, we are not going to take into account all its points, but only a few very particular ones.

As a period- $q \cdot 2^n$  saddle-node orbit undergoes a period doubling process there will be a control parameter value, prior to the duplication , at which the orbit will be a period- $q \cdot 2^n$  supercycle. Let  $R_{n,q}$  be such parameter, and let be

$$\left\{ \frac{1}{2}, f\left(\frac{1}{2}\right), \dots, f^{q \cdot 2^n - 1}\left(\frac{1}{2}\right) \right\} \quad (.1)$$

the supercycle in question.

We extract from supercycle the sequence

$$\{x_{m,q}\}_{m=1}^{m=2^n} = \left\{ x_{1,q} = \frac{1}{2}, x_{2,q} = f\left(\frac{1}{2}\right), x_{3,q} = f^2\left(\frac{1}{2}\right), \dots, x_{2^n,q} = f^{2^n-1}\left(\frac{1}{2}\right) \right\}$$

which we will name “restricted supercycle”. The restricted supercycle consist of  $2^n$  points, for every one of which the function  $f^{2^n}$  has a critical point close to the line  $y = x$ . The restricted supercycle is similar to the supercycle with which Feigenbaum works in the period doubling cascade, but it is different because there are  $q$  points of the supercycle (.1) around every critical point close to line  $y = x$ . Furthermore, the neighborhood of every critical point is visited  $q$  times if the order given by supercycle (.1) is followed. In other words, the graph of  $f^q$  is in the neighborhood of every point of a restricted supercycle. Accordingly, the scaling law of restricted supercycle gives the scaling law of the supercycle (.1)

What we are doing is to classify the  $q \cdot 2^n$  points of the saddle-node orbit in  $2^n$  sets, each one having  $q$  points. The  $2^n$  sets correspond to the  $2^n$  critical points of  $f^{2^n}$  closest to line  $y = x$ . The neighborhood of each one of these critical points captures  $q$  saddle-node points, in other words, the graph of  $f^q$  is captured in every one of such neighborhoods; for instance, in  $f^{3 \cdot 2^2}$  we notice that the graph of  $f^3$  is captured in the neighborhoods of the  $2^2$  critical points of  $f^{2^2}$  (Fig. .4).

Let's denote the distance from the  $m$ -th element of a  $2^n$ -restricted-supercycle to its nearest neighbor by

$$d_{n,q}(m) = x_{m,q} - f_{R_{n,q}}^{2^{n-1}}(x_{m,q})$$

Let's define the scaling (see Eq. (56) of [12]) by

$$\sigma_{n,q}(m) = \frac{d_{n+1,q}(m)}{d_{n,q}(m)}$$

Bearing in mind that  $x_m = f_{R_{n,q}}^m(0)$ , if we set  $m = 2^{n-i} - 1 \ll i \ll n$  then  $\sigma_{n,q}(2^{n-i})$  can be approximated by (see 57 of [12])

$$\sigma_{n,q}(2^{n-i}) \sim \frac{g_{i+1,q}(0) - g_{i+1,q}[(-\alpha)^{-i}g_{1,q}(0)]}{g_{i,q}(0) - g_{i,q}[(-\alpha)^{-i+1}g_{1,q}(0)]}$$

where  $g_{i,q}$  are the functions defined in [10].

The new variable

$$t_n(m) = \frac{m}{2^n}$$

or

$$t_n(2^{n-i}) = 2^{-i}$$

rescales the axis of iterates in such a way that all  $2^{n+1}$  iterates are within a unit interval.

Defining

$\sigma_{,q}(t_n(m)) \sim \sigma_{n,q}(m)$  (as  $n \rightarrow \infty$ )  
it turns out that (see Eq. (60) of [12])

$$\sigma_{,q}(-2^{-i-1}) = \frac{g_{i+1,q}(0) - g_{i+1,q}[(-\alpha)^{-i}g_{1,q}(0)]}{g_{i,q}(0) - g_{i,q}[(-\alpha)^{-i+1}g_{1,q}(0)]}$$

In the limit  $i \rightarrow \infty$  it yields

$$\sigma_{,q}(-2^{-i-1})_{i \rightarrow \infty} = \frac{g(0) - g[(-\alpha)^{-i}g_{1,q}(0)]}{g(0) - g[(-\alpha)^{-i+1}g_{1,q}(0)]} = \frac{1}{\alpha^2}$$

where

$$g_{i+1,q}(x) \rightarrow_{i \rightarrow \infty} g(x)$$

has been used (see [10]), and besides  $g(x)$  has a quadratic maximum, so

$$g[(-\alpha)^{-i}g_{1,q}(0)] \simeq g(0) + \frac{1}{2}g''(0) \cdot (-\alpha)^{-2i}g_{1,q}^2(0)$$

Let's notice that the scaling law does not depend on  $q$ , so we can drop this label and simply write

$$\sigma(-2^{-i-1})_{i \rightarrow \infty} = \frac{g(0) - g[(-\alpha)^{-i}g_{1,q}(0)]}{g(0) - g[(-\alpha)^{-i+1}g_{1,q}(0)]} = \frac{1}{\alpha^2}$$

The independence of  $q$  is critical to expound that the whole set of Saddle-Node bifurcations (for any  $q$ ) scales as the set of pitchfork bifurcation described by Feigenbaum. Once  $\sigma(-2^{-i-1})$  has been calculated, Feigenbaum [12] gives numbers in binary expression and demonstrates that  $|\sigma|$  behaves as  $\sim \frac{1}{\alpha}$  half the time and as  $\sim \frac{1}{\alpha^2}$  the other half.

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Fig. .1. Saddle-Node bifurcation genesis for  $f^3$ :  $r < r_c$  pre-bifurcation (thin dashed line),  $r = r_c$  at bifurcation (thick dashed line, tangent to  $y = x$ ),  $r > r_c$  post-bifurcation (dotted line).

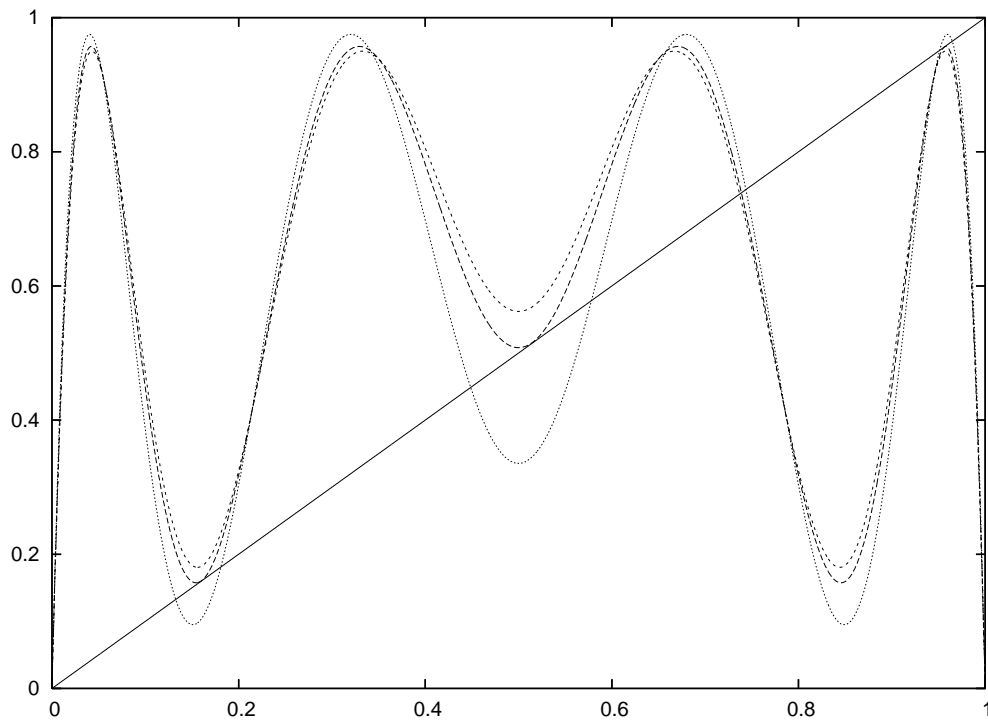


Fig. .2. Thick line shows that many iterations are necessary to cross the channels near a Saddle-Node bifurcation point.

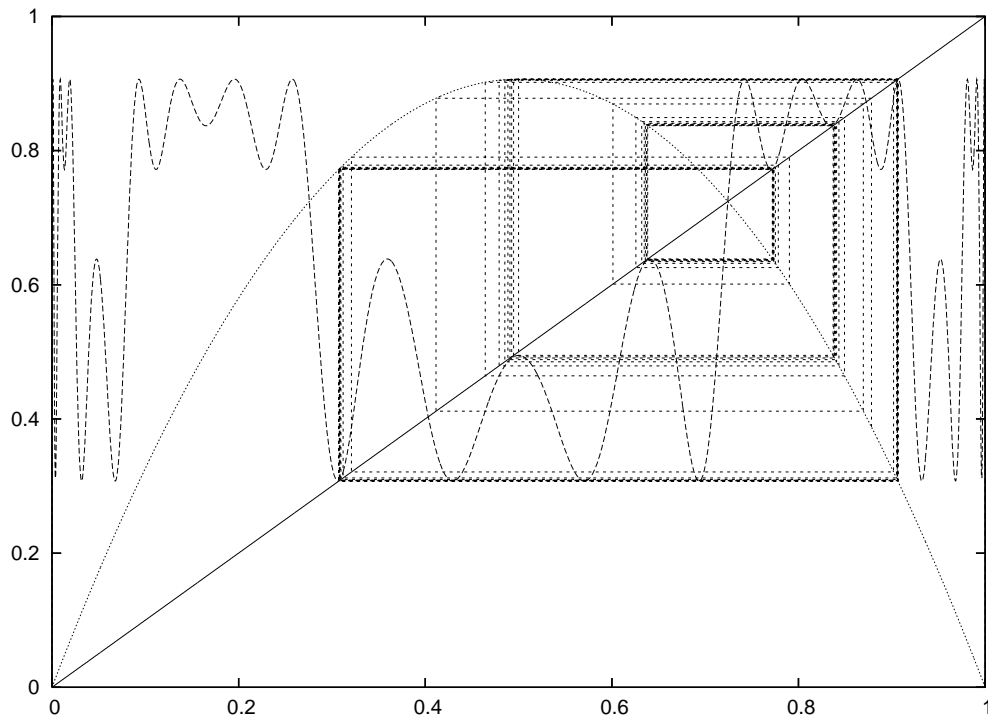


Fig. .3. Graph of  $f^{3 \cdot 2}$  (thin dashed line), at Saddle-Node bifurcation, reproduces the graph of  $f^3$  (see Fig. .1) around  $f^2$  (thick dashed line) maxima and minimum.

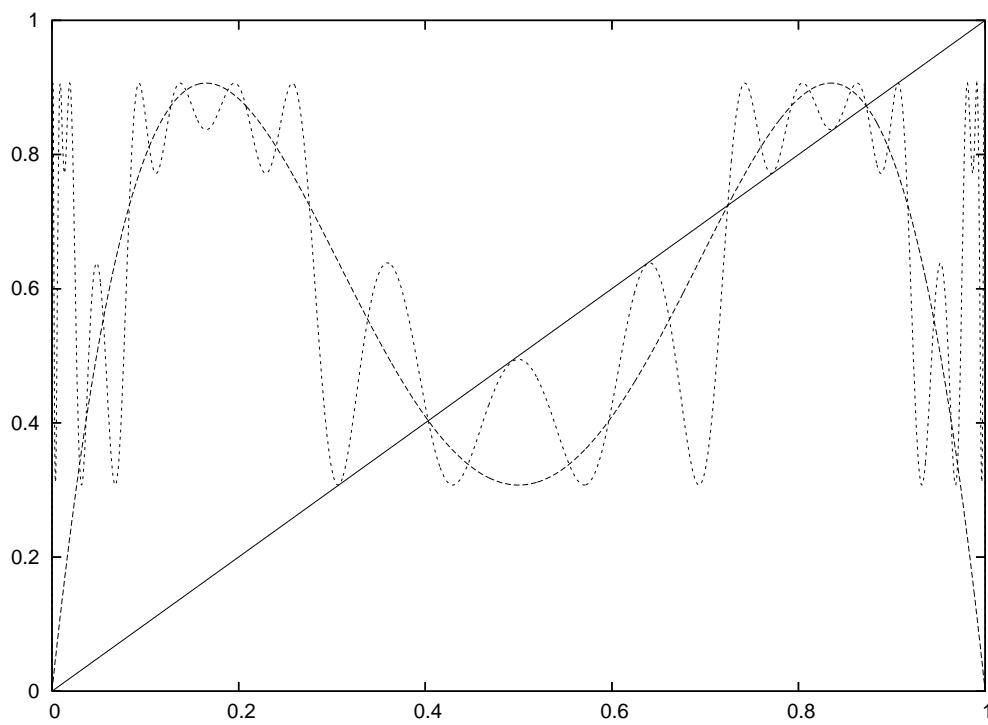




Fig. .4. Graph of  $f^{3 \cdot 2^2}$  (thin dashed line), at Saddle-Node bifurcation, reproduces the graph of  $f^3$  (see Fig. .1) around  $f^{2^2}$  (thick dashed line) maxima and minimum.

